

Finite amplitude inhomogeneous waves in Mooney-Rivlin viscoelastic solids.

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Abstract

New exact solutions are exhibited within the framework of finite viscoelasticity. More precisely, the solutions correspond to finite-amplitude, transverse, linearly-polarized, inhomogeneous motions superposed upon a finite homogeneous static deformation. The viscoelastic body is composed of a Mooney-Rivlin viscoelastic solid, whose constitutive equation consists in the sum of an elastic part (Mooney-Rivlin hyperelastic model) and a viscous part (Newtonian viscous fluid model). The analysis shows that the results are similar to those obtained for the purely elastic case; inter alia, the normals to the planes of constant phase and to the planes of constant amplitude must be orthogonal and conjugate with respect to the \mathbf{B} -ellipsoid, where \mathbf{B} is the left Cauchy-Green strain tensor associated with the initial large static deformation. However, when the constitutive equation is specialized either to the case of a neo-Hookean viscoelastic solid or to the case of a Newtonian viscous fluid, a greater variety of solutions arises, with no counterpart in the purely elastic case. These solutions include travelling inhomogeneous finite-amplitude damped waves and standing damped waves.

1 Introduction

Hayes and Saccomandi [1] initiated the study of finite amplitude waves in a simple class of viscoelastic materials of the differential type with a special constitutive equation for the Cauchy stress tensor. For these materials, denoted as *Mooney-Rivlin viscoelastic materials*, the Cauchy stress tensor is split into an elastic part, which coincides with the Cauchy stress tensor for a Mooney-Rivlin elastic material, and a dissipative part, which coincides with the Cauchy stress tensor for a Newtonian fluid (linear in the stretching tensor). Coleman and Noll [2] were the first to investigate the possibility of splitting the Cauchy stress into an elastic part and a dissipative part, and Fosdick and collaborators (Fosdick and Yu [3], Fosdick, Ketema, and Yu [4]) also considered Mooney-Rivlin viscoelastic materials.

With respect to wave propagation, Hayes and Saccomandi [1] obtained all homogeneous plane waves that may propagate in Mooney-Rivlin viscoelastic materials maintained in a static state of pure homogeneous deformation. Then they showed [5] that principal transverse homogeneous waves may propagate when superimposed on a special class of pseudo-plane inhomogeneous steady motions. Finally, they considered antiplane shear waves superimposed on a static biaxial stretch [6].

These results [1, 5, 6] provide a formidable corpus of exact solutions that may be useful as benchmarks for more complicated problems. Moreover, these solutions allow for a better understanding of the complex subject of nonlinear viscoelasticity [7]. The aim of this note is to extend these results to the case of *inhomogeneous* plane waves that is, plane waves for which the planes of constant phase are not parallel to the planes of constant amplitude. Here we show that finite amplitude, linearly polarized, transverse, inhomogeneous plane waves superimposed on a generic homogeneous deformation are rare because two strict conditions must be met: first, the planes of constant phase must be orthogonal to the planes of constant amplitude; second, the equations of motion lead to a set of two differential equations to be satisfied simultaneously. However, when the constitutive equation is specialized from Mooney-Rivlin to neo-Hookean viscoelastic materials and further, to viscous Newtonian fluids, then these strict conditions disappear and many more exact solutions are obtained, including finite-amplitude inhomogeneous damped plane waves. A remarkable feature of the solutions is that although the static strains are finite, the viscoelasticity theory nonlinear, and the waves of a generic (separable) finite amplitude inhomogeneous

travelling type, the governing equations nevertheless eventually reduce to a set of *linear* differential equations which are solved exactly. No small parameters nor asymptotic expansions are required, in contrast with most studies of finite-amplitude waves in solids (see Norris [8] for a review). The only restrictions lie with the specificity of the constitutive equation and with the form of the finite-amplitude inhomogeneous plane waves. This work follows the path laid out by Hadamard [9] for small-amplitude homogeneous plane waves in finitely deformed compressible elastic materials; by John [10] and by Currie and Hayes [11] for finite-amplitude homogeneous waves in deformed elastic compressible “Hadamard” materials and incompressible Mooney-Rivlin materials, respectively; and by Destrade [12] for finite-amplitude inhomogeneous waves in deformed elastic Mooney-Rivlin materials.

The plan of the paper is the following. In the next Section we introduce the basic equations governing the constitutive model of a Mooney-Rivlin viscoelastic material, and the propagation of finite amplitude plane inhomogeneous waves superposed on a large static pure homogeneous deformation. In Section 3, the equations of motion are solved and we find the conditions for the directions of polarization, of propagation, and of attenuation under which the waves may propagate. Sections 4 and 5 are devoted to the special cases of neo-Hookean viscoelastic materials and of Newtonian viscous fluids.

2 Preliminaries

2.1 Constitutive equations

The elastic part of the model is characterized by the Mooney-Rivlin strain-energy density W which, measured per unit volume in the undeformed state, is given by

$$2W = C(I - 3) + D(II - 3), \quad (2.1)$$

where the constants C, D satisfy $C \geq 0, D > 0$ or $C > 0, D \geq 0$ [13]. The sum $C + D$ is the infinitesimal shear modulus and I, II denote the first and second principal invariants of the left Cauchy-Green strain tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$:

$$I = \text{tr } \mathbf{B}, \quad 2II = (\text{tr } \mathbf{B})^2 - \text{tr}(\mathbf{B}^2). \quad (2.2)$$

The components of the gradient of deformation \mathbf{F} are

$$F_{iA} = \frac{\partial x_i}{\partial X_A}, \quad (2.3)$$

where x_i ($i = 1, 2, 3$) are the coordinates at time t of the point whose coordinates are X_i in the undeformed reference configuration. The Mooney-Rivlin viscoelastic solid is incompressible, which means that only isochoric deformations are possible; in other words, the condition

$$\det \mathbf{F} = 1, \quad (2.4)$$

must be satisfied at all times.

We assume that \mathbf{T}^D , the dissipative part of the stress, is given by

$$\mathbf{T}^D = \nu(\mathbf{L} + \mathbf{L}^T), \quad \mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}, \quad (2.5)$$

where $\nu > 0$ is a constant. The *Mooney-Rivlin viscoelastic constitutive equation* for the Cauchy stress tensor \mathbf{T} is given by

$$\mathbf{T} = -p\mathbf{1} + C\mathbf{B} - D\mathbf{B}^{-1} + \nu(\mathbf{L} + \mathbf{L}^T), \quad (2.6)$$

where p is the indeterminate pressure introduced by the incompressibility constraint (2.4), to be determined from the equations of motion and eventual boundary conditions.

We recall that in the absence of body forces, the equations of motion are

$$\operatorname{div} \mathbf{T} = \rho \ddot{\mathbf{x}}, \quad (2.7)$$

where ρ is the constant mass density of the material.

2.2 Transverse inhomogeneous motions in a homogeneously deformed material

Consider a finite static isochoric homogeneous deformation defined by

$$\mathbf{x} = \mathbf{F}\mathbf{X}, \quad (2.8)$$

where the F_{iA} are constant and $\det \mathbf{F} = 1$. For the deformation (2.8) both \mathbf{B} and \mathbf{B}^{-1} are constant and $\mathbf{L} = \mathbf{0}$. Then the following constant Cauchy stress tensor,

$$\mathbf{T} = -p_0\mathbf{1} + C\mathbf{B} - D\mathbf{B}^{-1}, \quad p_0 = \text{const.}, \quad (2.9)$$

clearly satisfies the equilibrium equations: $\operatorname{div} \mathbf{T} = \mathbf{0}$.

On this state of static deformation, superpose a finite motion taking the particle at \mathbf{x} to $\bar{\mathbf{x}}$, given by

$$\bar{\mathbf{x}} = \mathbf{x} + \mathbf{a}f(\mathbf{b} \cdot \mathbf{x})g(\mathbf{n} \cdot \mathbf{x} - vt). \quad (2.10)$$

Here f and g are functions to be determined and \mathbf{n} , \mathbf{b} , and \mathbf{a} are linearly independent unit vectors, such that $\mathbf{b} \times \mathbf{n} \neq \mathbf{0}$, $\mathbf{b} \cdot \mathbf{a} = \mathbf{n} \cdot \mathbf{a} = 0$. The motion (2.10) represents a transverse inhomogeneous plane wave, propagating with speed v in the direction of \mathbf{n} , linearly-polarized in the direction of \mathbf{a} . When the wave is not damped, the amplitude varies in the direction of \mathbf{b} ; when the wave is damped, the amplitude varies in some direction in the (\mathbf{b}, \mathbf{n}) -plane, to be determined from the equations of motion. The case where f is a constant (homogeneous waves) has already been considered by Hayes and Saccomandi [1]; the case of an *elastic* Mooney-Rivlin material ($\nu = 0$ in (2.6)) has been considered by Destrade [12].

For the motion (2.10), the deformation gradient $\bar{\mathbf{F}}$ is calculated using the chain rule as

$$\bar{\mathbf{F}} = \check{\mathbf{F}}\mathbf{F}, \quad \check{\mathbf{F}} := \mathbf{1} + f'g\mathbf{a} \otimes \mathbf{b} + fg'\mathbf{a} \otimes \mathbf{n}. \quad (2.11)$$

Taking the tensor product of $\check{\mathbf{F}}$ by $\mathbf{1} - f'g\mathbf{a} \otimes \mathbf{b} - fg'\mathbf{a} \otimes \mathbf{n}$, we find the identity tensor, and so

$$\bar{\mathbf{F}}^{-1} = \mathbf{F}^{-1}\check{\mathbf{F}}^{-1}, \quad \check{\mathbf{F}}^{-1} = \mathbf{1} - f'g\mathbf{a} \otimes \mathbf{b} - fg'\mathbf{a} \otimes \mathbf{n}. \quad (2.12)$$

It follows that the left Cauchy-Green tensor $\bar{\mathbf{B}}$ and its inverse $\bar{\mathbf{B}}^{-1}$ are given by

$$\bar{\mathbf{B}} = \check{\mathbf{F}}\mathbf{B}\check{\mathbf{F}}^T, \quad \bar{\mathbf{B}}^{-1} = \check{\mathbf{F}}^{-T}\mathbf{B}^{-1}\check{\mathbf{F}}^{-1}, \quad (2.13)$$

where $\check{\mathbf{F}}^{-T}$ is a shorthand notation for $(\check{\mathbf{F}}^{-1})^T$. Also, the time derivative of $\bar{\mathbf{F}}$ is given by

$$\dot{\bar{\mathbf{F}}} = \dot{\check{\mathbf{F}}}\mathbf{F}, \quad \text{where} \quad \dot{\check{\mathbf{F}}} = -v(f'g'\mathbf{a} \otimes \mathbf{b} - fg''\mathbf{a} \otimes \mathbf{n}). \quad (2.14)$$

Hence, the velocity gradient $\bar{\mathbf{L}}$ is here

$$\bar{\mathbf{L}} = \dot{\check{\mathbf{F}}} \bar{\mathbf{F}}^{-1} = \dot{\check{\mathbf{F}}} \check{\mathbf{F}}^{-1} = -v[f'g'\mathbf{a} \otimes \mathbf{b} + fg''\mathbf{a} \otimes \mathbf{n}] = \dot{\check{\mathbf{F}}}. \quad (2.15)$$

Now the corresponding Cauchy stress tensor $\bar{\mathbf{T}}$ (say) takes the form,

$$\bar{\mathbf{T}} = -\bar{p}\mathbf{1} + C\bar{\mathbf{B}} - D\bar{\mathbf{B}}^{-1} + \nu(\bar{\mathbf{L}} + \bar{\mathbf{L}}^T), \quad (2.16)$$

where the scalar \bar{p} is to be determined from the equations of motion.

2.3 Equations of motion

In order to write the equations of motion, it proves practical to introduce the Piola-Kirchhoff stress tensor $\bar{\mathbf{P}}$ associated with the motion (2.10) and defined when the static state of pure homogeneous finite deformation is considered to be the reference configuration. Hence,

$$\bar{P}_{ik} := \bar{T}_{ij} \check{F}_{kj}^{-1} = -\bar{p} \check{F}_{ki}^{-1} + C \check{F}_{ip} B_{pk} - D \bar{B}_{ij}^{-1} \check{F}_{kj}^{-1} + \nu \dot{\check{F}}_{ij} \check{F}_{kj}^{-1} + \nu \dot{\check{F}}_{ki}. \quad (2.17)$$

and the equations of motion read

$$\rho \partial^2 \bar{x}_i / \partial t^2 = \partial \bar{P}_{ik} / \partial x_k. \quad (2.18)$$

We introduce the real variables

$$\zeta := \mathbf{b} \cdot \mathbf{x}, \quad \eta := \mathbf{n} \cdot \mathbf{x}, \quad \text{so that} \quad f = f(\zeta), \quad g = g(\eta - vt). \quad (2.19)$$

Now we compute in turn the terms of (2.18), using the expansion (2.17)₂ for \bar{P}_{ik} . First, the left hand-side term,

$$\rho \partial^2 \bar{x}_i / \partial t^2 = \rho v^2 f g'' a_i. \quad (2.20)$$

Next, the first term on the right hand-side,

$$-\partial(\bar{p} \check{F}_{ki}^{-1}) / \partial x_k = -(\partial \bar{p} / \partial x_k) \check{F}_{ki}^{-1} = -(\bar{p}_{,\zeta} b_k + \bar{p}_{,\eta} n_k) \check{F}_{ki}^{-1} = -\bar{p}_{,\zeta} b_i - \bar{p}_{,\eta} n_i, \quad (2.21)$$

where we used the Euler-Jacobi-Piola (EJP) identity for incompressible materials $\partial \check{F}_{ki}^{-1} / \partial x_k = 0$ in the first equality, and $a_k n_k = a_k b_k = 0$ in the last.

Now, the second term on the right hand-side,

$$\begin{aligned} \partial(\check{F}_{ip} B_{pk}) / \partial x_k &= (\partial \check{F}_{ip} / \partial x_k) B_{pk} \\ &= [f'' g(\mathbf{b} \cdot \mathbf{Bb}) + 2f' g'(\mathbf{n} \cdot \mathbf{Bb}) + f g''(\mathbf{n} \cdot \mathbf{Bn})] a_i. \end{aligned} \quad (2.22)$$

Then, the expansion of the third term on the right hand-side requires the EJP identity,

$$\begin{aligned} \partial(\bar{B}_{ij}^{-1} \check{F}_{kj}^{-1}) / \partial x_k &= (\partial \bar{B}_{ij}^{-1} / \partial x_k) \check{F}_{kj}^{-1} = (\partial \bar{B}_{ij}^{-1} / \partial x_k) \delta_{kj} = \partial \bar{B}_{ij}^{-1} / \partial x_j \\ &= -[f'' g(\mathbf{a} \cdot \mathbf{B}^{-1} \mathbf{b}) + f' g'(\mathbf{n} \cdot \mathbf{B}^{-1} \mathbf{n})] b_i \\ &\quad - [f' g'(\mathbf{a} \cdot \mathbf{B}^{-1} \mathbf{b}) + f g''(\mathbf{n} \cdot \mathbf{B}^{-1} \mathbf{n})] n_i \\ &\quad - [f'' g + 2f' g'(\mathbf{n} \cdot \mathbf{b}) + f g''] B_{ij}^{-1} a_j \\ &\quad + [2f' f'' g^2 + (3f'^2 + f f'') g g'(\mathbf{n} \cdot \mathbf{b}) + f f'(g'^2 + g g'')](\mathbf{a} \cdot \mathbf{B}^{-1} \mathbf{a}) b_i \\ &\quad + [2f^2 g' g'' + f f'(3g'^2 + g g'')(\mathbf{n} \cdot \mathbf{b}) + (f'^2 + f f'') g g'](\mathbf{a} \cdot \mathbf{B}^{-1} \mathbf{a}) n_i. \end{aligned} \quad (2.23)$$

The expansion of the fourth term on the right hand-side also involves the EJP identity,

$$\begin{aligned}
\partial(\dot{\check{F}}_{ij}^{-1}\check{F}_{kj}^{-1})/\partial x_k &= (\partial\dot{\check{F}}_{ij}^{-1}/\partial x_k)\check{F}_{kj}^{-1} \\
&= -v[f''g'b_kb_j + f'g''(n_kb_j + b_kn_j) + fg'''n_kn_j]a_i\check{F}_{kj}^{-1} \\
&= -v[f''g' + 2f'g''(\mathbf{n} \cdot \mathbf{b}) + fg''']a_i.
\end{aligned} \tag{2.24}$$

Finally the fifth and last term on the right hand-side turns out to be zero:

$$\partial\dot{\check{F}}_{ki}^{-1}/\partial x_k = -v[f''g'b_ka_kb_i + f'g''n_ka_kb_i + f'g''b_ka_kn_i + fg'''n_ka_kn_i] = 0, \tag{2.25}$$

because $a_kn_k = a_kb_k = 0$.

Now we proceed to the resolution of the equations of motion (2.18) using (2.17) and (2.20)-(2.25). We treat in turn the cases of a Mooney-Rivlin viscoelastic material ($C > 0, D > 0, \nu > 0$), of a neo-Hookean viscoelastic material ($C > 0, D = 0, \nu > 0$), and of a Newtonian viscous fluid ($C = 0, D = 0, \nu > 0$).

3 Mooney-Rivlin viscoelastic solid

3.1 Orthogonality of \mathbf{n} and \mathbf{b}

Destrade [12] proved that finite-amplitude inhomogeneous plane waves of evanescent sinusoidal type may propagate in a homogeneously deformed Mooney-Rivlin *elastic* material only when the planes of constant phase are orthogonal to the planes of constant amplitude. Here we prove that this result is not affected by the addition in (2.6) of a *viscous* part to the Cauchy stress.

Take the superposed motion $\mathbf{a}f(\mathbf{b} \cdot \mathbf{x})g(\mathbf{n} \cdot \mathbf{x} - vt)$ in (2.10) to be of evanescent sinusoidal type,

$$f = \alpha e^{-\omega\sigma\zeta}, \quad g = 2\beta \cos \omega v^{-1}(\eta - vt), \tag{3.1}$$

where the amplitudes α, β , the frequency ω , and the attenuation factor σ are arbitrary real scalars. Then, leaving aside the pressure terms (2.21) for the time being, we find by inspection of (2.20) and (2.22)-(2.25) that all the other terms appearing in the equations of motion (2.18) can be decomposed along the set of linearly independent functions $e^{-\omega\sigma\zeta} \cos \omega v^{-1}(\eta - vt)$,

$e^{-\omega\sigma\zeta} \sin \omega v^{-1}(\eta - vt)$, $e^{-2\omega\sigma\zeta} \cos 2\omega v^{-1}(\eta - vt)$, $e^{-2\omega\sigma\zeta} \sin 2\omega v^{-1}(\eta - vt)$, and $e^{-2\omega\sigma\zeta}$. The only terms along this latter (time-independent) function come from the decomposition (2.23), where we have

$$2f'f''g^2 + ff'g'^2 = -2\alpha^2\beta^2\omega^3\sigma(2\sigma^2 + v^{-2})e^{-2\omega\sigma\zeta} + \dots$$

$$\text{and } ff'(3g'^2 + gg'') = -4\alpha^2\beta^2\omega^3\sigma v^{-2}e^{-2\omega\sigma\zeta} + \dots \quad (3.2)$$

Here the ellipses stand for terms proportional to time-dependent functions. It follows from the equations of motion that \bar{p} must also be of this form,

$$\bar{p} = -\alpha^2\beta^2\omega^2 p_1 e^{-2\omega\sigma\zeta} + \dots, \quad (3.3)$$

where p_1 is constant. Then the time-independent part of the equations of motion (2.18) is simply

$$\mathbf{0} = [-p_1 + D(2\sigma^2 + v^{-2})(\mathbf{a} \cdot \mathbf{B}^{-1} \mathbf{a})]\mathbf{b} + 2Dv^{-2}(\mathbf{n} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{B}^{-1} \mathbf{a})\mathbf{n}. \quad (3.4)$$

Because \mathbf{n} and \mathbf{b} cannot be parallel for the motion (2.10) to be inhomogeneous, we conclude that $p_1 = D(2\sigma^2 + v^{-2})(\mathbf{a} \cdot \mathbf{B}^{-1} \mathbf{a})$ and that

$$\mathbf{n} \cdot \mathbf{b} = 0. \quad (3.5)$$

This important result, established in [12] for a Mooney-Rivlin elastic solid, still holds when the solid is viscoelastic with a constitutive equation of the form (2.6) because as we just saw, the viscous terms (2.24) contribute to the equations of motion only with time-dependent functions. Turning back to a general inhomogeneous motion of the form (2.10), this result leads us to assume that here also, \mathbf{n} and \mathbf{b} are orthogonal, so that $(\mathbf{n}, \mathbf{b}, \mathbf{a})$ is an orthonormal basis. Accordingly, some simplifications occur in the equations of motion.

We introduce the notation

$$\rho v_n^2 := C(\mathbf{n} \cdot \mathbf{B} \mathbf{n}) + D(\mathbf{a} \cdot \mathbf{B}^{-1} \mathbf{a}) > 0,$$

$$\rho v_b^2 := C(\mathbf{b} \cdot \mathbf{B} \mathbf{b}) + D(\mathbf{a} \cdot \mathbf{B}^{-1} \mathbf{a}) > 0,$$

$$\rho b := C(\mathbf{n} \cdot \mathbf{B} \mathbf{b}). \quad (3.6)$$

Here v_n (v_b) is the speed of an homogeneous finite-amplitude plane wave propagating in the direction of \mathbf{n} (\mathbf{b}) and polarized in the direction of \mathbf{a} , as

proved by Boulanger and Hayes [13]. Then the projection of the equation of motion (2.18) along \mathbf{a} is written in compact form as

$$\rho v_b^2 f'' g + 2\rho b f' g' - \nu v(f g''' + f'' g') + \rho(v_n^2 - v^2) f g'' = 0. \quad (3.7)$$

Finally, the projections of the equations of motion (2.18) along \mathbf{n} and along \mathbf{b} yield expressions for the derivatives $\bar{p}_{,\eta}$ and $\bar{p}_{,\zeta}$; writing $\bar{p}_{,\eta\zeta} = \bar{p}_{,\eta\zeta}$ gives [12],

$$\begin{aligned} (f''' g + f' g'')(\mathbf{a} \cdot \mathbf{B}^{-1} \mathbf{n}) - (f'' g' + f g''')(\mathbf{a} \cdot \mathbf{B}^{-1} \mathbf{b}) \\ + [(f' f'' - f f''') g g' - (g' g'' - g g''') f f'](\mathbf{a} \cdot \mathbf{B}^{-1} \mathbf{a}) = 0. \end{aligned} \quad (3.8)$$

Hence the propagation of finite-amplitude, linearly polarized, transverse, plane waves in a deformed Mooney-Rivlin viscoelastic solid is governed by two equations: the “balance equation” (3.7) and the “compatibility equation” (3.8), to be solved simultaneously.

3.2 Derivation of the solutions

First divide (3.7) by $f(\zeta)g'(\eta - vt)$ to get

$$\rho v_b^2 \frac{f''}{f} \frac{g}{g'} + 2\rho b \frac{f'}{f} - \nu v \left(\frac{g'''}{g'} + \frac{f''}{f} \right) + \rho(v_n^2 - v^2) \frac{g''}{g'} = 0. \quad (3.9)$$

Then differentiate first with respect to the argument of f , next with respect to the argument of g , to obtain

$$(f''/f)'(g/g')' = 0, \quad (3.10)$$

so that either f''/f or g/g' is a constant. The following cases cover all possibilities,

$$(i) f''/f = -k_1^2, \quad (ii) f''/f = k_1^2, \quad (iii) g'/g = k_2. \quad (3.11)$$

Without loss of generality, we pick $k_1 > 0$ in Case (i) and in Case (ii); in Case (iii) we take $k_2 > 0$ to avoid solutions which grow exponentially with time t (note that the forthcoming analysis can also be conducted to accomodate solutions which do grow exponentially with time). We treat each case in turn, keeping in mind that f must be trigonometric when g is exponential and *vice-versa*, for the motion to be inhomogeneous [12].

Case (i): $f(\zeta) = A \cos k_1 \zeta + B \sin k_1 \zeta$.

Substitution of this form of solution into the balance equation (3.9) and differentiation with respect to ζ yields $\rho b(f'/f)' = 0$ and so,

$$\mathbf{n} \cdot \mathbf{B}\mathbf{b} = 0. \quad (3.12)$$

This condition means that the unit vectors \mathbf{n} and \mathbf{b} must be conjugate with respect to the central elliptical section of the \mathbf{B} -ellipsoid, $\mathbf{x} \cdot \mathbf{B}\mathbf{x} = 1$, by the plane orthogonal to \mathbf{a} . Being orthogonal, they must be along the principal axes of this ellipse.

Then the balance equation (3.7) (with $b = 0$) gives a third order linear differential equation for g ,

$$\nu v g''' - \rho(v_n^2 - v^2)g'' - k_1^2 \nu v g' + k_1^2 \rho v_b^2 g = 0. \quad (3.13)$$

On the other hand, the compatibility equation (3.8) written as an identity for $\cos k_1 \zeta$, $\sin k_1 \zeta$, $\cos 2k_1 \zeta$, and $\sin 2k_1 \zeta$ yields

$$g'g'' - gg''' = 0, \quad (g'' - k_1^2 g)(\mathbf{a} \cdot \mathbf{B}^{-1}\mathbf{n}) = 0, \quad (g''' - k_1^2 g')(\mathbf{a} \cdot \mathbf{B}^{-1}\mathbf{b}) = 0. \quad (3.14)$$

These three equations are satisfied simultaneously either when (a) $g'' = k_1^2 g$ or when (b) $g''/g = \text{const.}$, $\mathbf{a} \cdot \mathbf{B}^{-1}\mathbf{n} = \mathbf{a} \cdot \mathbf{B}^{-1}\mathbf{b} = 0$.

In *Case (ia)*, g is an exponential function. Discarding solutions which blow up with time, we find

$$g(\eta - vt) = e^{k_1(\eta - vt)}. \quad (3.15)$$

Substitution into the third-order differential equation (3.13) fixes v as

$$\rho v^2 = \rho(v_n^2 - v_b^2) = C[(\mathbf{n} \cdot \mathbf{B}\mathbf{n}) - (\mathbf{b} \cdot \mathbf{B}\mathbf{b})]. \quad (3.16)$$

The quantity v is real when \mathbf{n} and \mathbf{b} are in the respective directions of the minor and major axes of the elliptical section of the \mathbf{B} -ellipsoid by the plane orthogonal to \mathbf{a} .

In *Case (ib)*, g is of the form

$$g(\eta - vt) = e^{k_2(\eta - vt)}, \quad (3.17)$$

where $k_2 > 0$ to avoid blowing-up solutions. The conditions on \mathbf{n} , \mathbf{b} , \mathbf{a} , together with (3.12), imply that these unit vectors are along *principal directions* of the \mathbf{B} -ellipsoid. Substitution into the third-order differential equation (3.13) relates k_1 to k_2 through

$$k_1^2 = \frac{\rho(v_n^2 - v^2) - k_2 \nu v}{\rho v_b^2 - k_2 \nu v} k_2^2. \quad (3.18)$$

Recall that k_1 is assumed real so that, in order to construct a solution here, we may pick any values for $k_2 > 0$ and for $v > 0$, as long as $\rho(v_n^2 - v^2) - k_2\nu v$ and $\rho v_b^2 - k_2\nu v$ are of the same sign, where v_n, v_b are given by (3.6) with $\mathbf{n}, \mathbf{b}, \mathbf{a}$ along principal directions of the \mathbf{B} -ellipsoid. Then k_1 is given by (3.18).

Case (ii): $f(\zeta) = A \cosh k_1 \zeta + B \sinh k_1 \zeta$.

Substitution of this form of solution into the balance equation (3.9) and differentiation with respect to ζ yields the condition (3.12) as in *Case (i)*.

Then the balance equation (3.7) (with $b = 0$) gives a third order linear differential equation for g ,

$$\nu v g''' - \rho(v_n^2 - v^2)g'' + k_1^2 \nu v g' - k_1^2 \rho v_b^2 g = 0. \quad (3.19)$$

Also, the compatibility equation (3.8) written as an identity for $\cosh k_1 \zeta$, $\sinh k_1 \zeta$, $\cosh 2k_1 \zeta$, and $\sinh 2k_1 \zeta$ yields

$$g'g'' - gg''' = 0, \quad (g'' + k_1^2 g)(\mathbf{a} \cdot \mathbf{B}^{-1} \mathbf{n}) = 0, \quad (g''' + k_1^2 g')(\mathbf{a} \cdot \mathbf{B}^{-1} \mathbf{b}) = 0. \quad (3.20)$$

These three equations are satisfied simultaneously either when (a) $g'' = -k_1^2 g$ or when (b) $g''/g = \text{const.}$, $\mathbf{a} \cdot \mathbf{B}^{-1} \mathbf{n} = \mathbf{a} \cdot \mathbf{B}^{-1} \mathbf{b} = 0$.

In *Case (iia)*, g is of the form

$$g(\eta - vt) = d_1 \cos k_1(\eta - vt) + d_2 \sin k_1(\eta - vt), \quad (3.21)$$

and substitution into the third-order differential equation (3.19) fixes v again as (3.16), $v^2 = v_n^2 - v_b^2$.

In *Case (iib)*, g is of the form $g = d_1 \cos k_2(\eta - vt) + d_2 \sin k_2(\eta - vt)$, where k_2 is arbitrary. The conditions on $\mathbf{n}, \mathbf{b}, \mathbf{a}$, together with (3.12) imply that these unit vectors are along principal directions. Then substitution of g into the third-order differential equation (3.19) gives

$$\nu v(k_1^2 - k_2^2)g' - \rho[k_1^2 v_b^2 - k_2^2(v_n^2 - v^2)]g = 0. \quad (3.22)$$

This equation is satisfied for g trigonometric and $\nu \neq 0$ only when the respective coefficients of g and g' are zero, conditions which lead back to a solution of the form (3.21) and v uniquely determined by (3.16). The existence of these ‘special principal motions’ is therefore more limited than when the solid is purely elastic [12] because there they may propagate at an arbitrary speed v within the interval $[0, v_n]$.

Case (iii): $g(\eta - vt) = e^{k_2(\eta - vt)}$.

Substituting this form of solution into the balance equation (3.7), we obtain a second-order linear differential equation for f ,

$$(\rho v_b^2 - k_2 \nu v) f'' + 2k_2 \rho b f' + k_2^2 [\rho(v_n^2 - v^2) - k_2 \nu v] f = 0. \quad (3.23)$$

Also, the compatibility equation (3.8) reduces to

$$(f''' + k_2^2 f')(\mathbf{a} \cdot \mathbf{B}^{-1} \mathbf{n}) - k_2(f'' + k_2^2 f)(\mathbf{a} \cdot \mathbf{B}^{-1} \mathbf{b}) + k_2(f' f'' - f f''')(\mathbf{a} \cdot \mathbf{B}^{-1} \mathbf{a}) e^{k_2(\eta - vt)} = 0. \quad (3.24)$$

Differentiating this equation with respect to $(\eta - vt)$, we find that $f''/f = \text{const}$. This constant must be negative so that the combination of f and g represents an inhomogeneous motion [12]. But if f is trigonometric, then the second order differential equation (3.23) can be satisfied only when $\rho b = C(\mathbf{n} \cdot \mathbf{B} \mathbf{b}) = 0$, which leads us back to *Case (i)* and the solutions (3.15)-(3.16) (non-principal motions) and (3.17)-(3.18) (principal motions).

3.3 Summary of results and comparison with the purely elastic case

For a deformed *elastic* Mooney-Rivlin material, Destrade [12] proved that only two types of finite-amplitude inhomogeneous motions are possible: either f is trigonometric and g is exponential,

$$f(\zeta) = a_1 \cos k_1 \zeta + a_2 \sin k_1 \zeta, \quad g(\eta - vt) = d_1 e^{k_2(\eta - vt)}, \quad (3.25)$$

or f is hyperbolic and g is trigonometric,

$$f(\zeta) = a_1 \cosh k_1 \zeta + a_2 \sinh k_1 \zeta, \quad g(\eta - vt) = d_1 \cos k_2(\eta - vt) + d_2 \sin k_2(\eta - vt), \quad (3.26)$$

where a_1, a_2, d_1, d_2 are constants and k_2 is arbitrary.

When the vectors $(\mathbf{a}, \mathbf{b}, \mathbf{n})$ are not aligned with the principal axes of deformation, then the quantities k_1 and v are determined by

$$k_1 = k_2, \quad \rho v^2 = \rho(v_n^2 - v_b^2) = C[(\mathbf{n} \cdot \mathbf{B} \mathbf{n}) - (\mathbf{b} \cdot \mathbf{B} \mathbf{b})], \quad (3.27)$$

and the unit vectors \mathbf{n} and \mathbf{b} must be conjugate with respect to the \mathbf{B} -ellipsoid, $\mathbf{x} \cdot \mathbf{B} \mathbf{x} = 1$, that is the condition (3.12) must be satisfied.

When the vectors $(\mathbf{a}, \mathbf{b}, \mathbf{n})$ are aligned with the principal axes of deformation ('special principal motions'), then the quantity v may take *any* value within the interval $[0, v_n]$ and k_1 is given by

$$k_1^2 = \frac{v_n^2 - v^2}{v_b^2} k_2^2. \quad (3.28)$$

Here we saw that the corresponding results for a deformed *viscoelastic* Mooney-Rivlin material are essentially the same as above, with the following difference. When the three vectors $\mathbf{a}, \mathbf{b}, \mathbf{n}$ are not aligned with the principal axes of deformation, then the solution is indeed of the form (3.25)-(3.27) with the condition (3.12). However, when all three vectors $\mathbf{a}, \mathbf{b}, \mathbf{n}$ are aligned with the principal axes of deformation, then the solution is of the form (3.25) or (3.26) but: when f is trigonometric and g is exponential as in (3.25), then k_1 is given by (3.18), and v, k_2 are arbitrary as long as k_1^2 is positive; when f is hyperbolic and g is trigonometric as in (3.26), then v are determined by (3.16) that is, v is fixed and no longer arbitrary as in the elastic case.

Now we see that when $D = 0$, the visco-elastic solid allows more solutions than its elastic counterpart.

4 Neo-Hookean viscoelastic solid

When $D = 0$ ($C \neq 0$) the strain-energy density (2.1) reduces to the well-known neo-Hookean form. A great variety of exact solutions is uncovered, in particular solutions in the form of finite-amplitude damped inhomogeneous waves. Note that this problem was recently studied in the *elastic* compressible case by Rodrigues Ferreira and Boulanger [14].

We start with the equations of motion (2.18) written at $D = 0$. The unit vectors \mathbf{b} and \mathbf{n} are not necessarily orthogonal but the triad $(\mathbf{a}, \mathbf{b}, \mathbf{n})$ is composed of linearly independent vectors, so that the coefficients along each vector in the equations of motion are all zero. The coefficients along \mathbf{b} and \mathbf{n} are respectively,

$$0 = -\bar{p}_{,\zeta}, \quad 0 = -\bar{p}_{,\eta}, \quad (4.1)$$

from which we deduce that $\bar{p} = \text{const.}$ satisfies this part of the equations of motion. The other part, along \mathbf{a} , is independent of \bar{p} and reads

$$\rho(v^2 - v_n^2)fg'' - \rho v_b^2 f''g - 2\rho b f'g' + \nu v[f''g' + 2f'g''(\mathbf{n} \cdot \mathbf{b}) + fg'''] = 0, \quad (4.2)$$

where the quantities

$$\rho v_n^2 = C \mathbf{n} \cdot \mathbf{B} \mathbf{n} > 0, \quad \rho v_b^2 = C \mathbf{b} \cdot \mathbf{B} \mathbf{b} > 0, \quad \rho b = C \mathbf{n} \cdot \mathbf{B} \mathbf{b}, \quad (4.3)$$

were used.

Now divide (4.2) by $f(\zeta)g'(\eta - vt)$ and differentiate, first with respect to ζ and then with respect to $(\eta - vt)$, to obtain the necessary condition

$$\rho v_b^2 \left(\frac{f''}{f} \right)' \left(\frac{g}{g'} \right)' = 2\nu v (\mathbf{n} \cdot \mathbf{b}) \left(\frac{f'}{f} \right)' \left(\frac{g''}{g'} \right)'. \quad (4.4)$$

We separate the functions of different variables in this equation to conclude that either

$$\left(\frac{f'}{f} \right)' \left(\frac{g}{g'} \right)' = 0, \quad \text{or} \quad \rho v_b^2 \frac{(f''/f)'}{(f'/f)'} = 2\nu v (\mathbf{n} \cdot \mathbf{b}) \frac{(g''/g')'}{(g/g')'} = \text{const.} \quad (4.5)$$

The following cases cover all possibilities,

$$(i) f' = k_1 f, \quad (ii) g' = k_2 g, \quad (4.6)$$

and (iii) $\rho v_b^2 f'' - k_1 f' - k_3 f = 0$, $2\nu v (\mathbf{n} \cdot \mathbf{b}) g'' - k_2 g' - k_1 g = 0$, where k_1, k_2, k_3 are constants. In fact, using expressions for f'' , g'' , and g''' derived from (iii), and substituting into (4.2), it is straightforward to check that Case (iii) leads to Cases (i) or (ii).

In Case (i) we have

$$f(\zeta) = \exp(k_1 \zeta),$$

$$g''' + \left[\frac{\rho}{\nu v} (v^2 - v_n^2) + 2k_1 (\mathbf{n} \cdot \mathbf{b}) \right] g'' + k_1 (k_1 - 2 \frac{\rho}{\nu v} b) g' - \rho \frac{v_b^2}{\nu v} k_1^2 g = 0. \quad (4.7)$$

In Case (ii) we have

$$g(\eta - vt) = \exp[k_2 (\eta - vt)],$$

$$(\rho v_b^2 - k_1 \nu v) f'' + 2k_2 [\rho b - \nu v k_2 (\mathbf{n} \cdot \mathbf{b})] f' - k_2^2 [\rho (v^2 - v_n^2) + \nu v k_2] f = 0. \quad (4.8)$$

Now consider (4.7) in greater detail. The characteristic equation associated with the third-order differential equation for g is a cubic. Depending on the values of the coefficients (that is, on the choices for v and k_1), the cubic has either only real roots or one real root and two complex conjugate roots.

Real roots lead to an homogeneous motion and must be discarded. Pure imaginary roots lead to inhomogeneous plane waves with attenuation in the direction of \mathbf{b} and propagation in the direction of \mathbf{n} . Non-pure-imaginary complex roots lead to inhomogeneous *damped* plane waves with attenuation in the plane of \mathbf{b} and \mathbf{n} , exponential damping with time, and propagation in the direction of \mathbf{n} . Because v or k_1 may be prescribed a priori, there is an infinity of such solutions, of which we present one explicitly.

We choose v such that the coefficient of g'' in $(4.7)_2$ is zero, that is we fix v as

$$\rho v = \rho v_o := k_1 \nu (\mathbf{n} \cdot \mathbf{b}) + \sqrt{[k_1 \nu (\mathbf{n} \cdot \mathbf{b})]^2 + (\rho v_n)^2}. \quad (4.9)$$

Then (4.7) reduces to

$$g''' + \beta_o g' - \gamma_o g = 0, \quad \beta_o = k_1 \left(k_1 - 2 \frac{\rho b}{\nu v_o} \right), \quad \gamma_o = k_1^2 \frac{\rho v_b^2}{\nu v_o}. \quad (4.10)$$

Now we seek a solution in the form

$$g(\eta - v_o t) = e^{\lambda_o(\eta - v_o t)} \cos \omega_o(\eta - v_o t), \quad (4.11)$$

where λ_o and ω_o are real constants determined as follows. Substituting (4.11) into (4.10) , we find in turn that ω_o and λ_o satisfy

$$\omega_o^2 = 3\lambda_o^2 + \beta_o, \quad 8\lambda_o^3 + 2\beta_o\lambda_o + \gamma_o = 0. \quad (4.12)$$

The real root of the cubic in λ_o is

$$\lambda_o = \frac{1}{12} [12\sqrt{12\beta_o^3 + 81\gamma_o} - 108\gamma_o]^{\frac{1}{3}} - \beta_o [12\sqrt{12\beta_o^3 + 81\gamma_o} - 108\gamma_o]^{-\frac{1}{3}}. \quad (4.13)$$

That ω_o is real is ensured for instance when $\beta_o > 0$ (see Eq. $(4.12)_1$). A few lines of calculation show that a sufficient condition for β_o to be positive is

$$k_1^2 > \frac{(2\rho b)^2}{\nu^2[v_n^2 + 4b(\mathbf{n} \cdot \mathbf{b})]} = \frac{4\rho C(\mathbf{n} \cdot \mathbf{Bb})^2}{\nu^2[(\mathbf{n} \cdot \mathbf{Bn}) + 4(\mathbf{n} \cdot \mathbf{Bb})(\mathbf{n} \cdot \mathbf{b})]}. \quad (4.14)$$

Thus for any *arbitrary* value of k_1 satisfying this inequality, the following finite-amplitude damped inhomogeneous plane wave may propagate in a deformed neo-Hookean viscoelastic body,

$$\bar{\mathbf{x}} = \mathbf{x} + \alpha \mathbf{a} e^{(k_1 \mathbf{b} + \lambda_o \mathbf{n}) \cdot \mathbf{x} - \lambda_o v_o t} \cos \omega_o(\mathbf{n} \cdot \mathbf{x} - v_o t), \quad (4.15)$$

where \mathbf{x} is given by (2.8), α is arbitrary, the orientation of the triad of unit vectors $\mathbf{a}, \mathbf{b}, \mathbf{n}$ (\mathbf{b} and \mathbf{n} orthogonal to \mathbf{a}) is arbitrary, and λ_o, ω_o , and v_o are given by (4.13), (4.12)₁, and (4.9), respectively. The wave propagates in the direction of \mathbf{n} , is exponentially damped with time, and is attenuated in the direction of $k_1\mathbf{b} + \lambda_0\mathbf{n}$.

To conclude this Section, we point out that inspection of (4.4) makes it clear that a neo-Hookean viscoelastic constitutive equation yields more inhomogeneous waves than a neo-Hookean elastic constitutive equation, where $\nu = 0$.

5 Newtonian viscous fluids

With $C = D = 0$, equation (2.6) is the constitutive relation defining an incompressible Newtonian viscous fluid of viscosity ν , and (2.7), or their specialization (4.2) at $C = 0$, are the (Lagrangian) Navier-Stokes equations in the absence of body forces for the finite amplitude inhomogeneous plane wave (2.10), that is $f''g' + [2(\mathbf{n} \cdot \mathbf{b})f' + (\rho v/\nu)f]g'' + fg''' = 0$. Integrating with respect to the argument of g and taking the constant of integration to be zero for simplicity, we obtain

$$f''g + [2(\mathbf{n} \cdot \mathbf{b})f' + \frac{\rho v}{\nu}f]g' + fg'' = 0. \quad (5.1)$$

With methods similar to those presented in the two previous sections, it is a straightforward matter to show that either f or g are pure exponential functions that is, either

$$f(\zeta) = \exp(k_1\zeta), \quad \text{and} \quad g'' + [2k_1(\mathbf{n} \cdot \mathbf{b}) + \frac{\rho v}{\nu}]g' + k_1^2g = 0, \quad (5.2)$$

or

$$g(\eta - vt) = \exp[k_2(\eta - vt)], \quad \text{and} \quad f'' + 2k_2(\mathbf{n} \cdot \mathbf{b})f' + k_2(\frac{\rho v}{\nu} + k_2)f = 0. \quad (5.3)$$

where k_1, k_2 are arbitrary constants. The latter case represents a standing damped wave, whatever the form of f may be. The former is a travelling inhomogeneous damped wave as long as the characteristic equation associated with the second-order differential equation for g has complex roots. This condition reads

$$0 < \rho v < 2k_1\nu[1 - (\mathbf{n} \cdot \mathbf{b})] =: \rho v_0. \quad (5.4)$$

Hence the constants k_1 and v are *arbitrary* as long as k_1 is positive and v is less than v_0 defined above. The following displacement field \mathbf{u} is solution to the Navier-Stokes equations for any orientation of the pair of unit vectors \mathbf{b}, \mathbf{n} in the plane $\mathbf{a} \cdot \mathbf{x} = 0$ (which may be taken without loss of generality as $z = 0$ say, because the fluid is isotropic),

$$\begin{aligned} u_1 &= 0, & u_2 &= 0, \\ u_3 &= e^{k_1[\mathbf{b} \cdot \mathbf{x} + \sqrt{1-\kappa^2}(\mathbf{n} \cdot \mathbf{x} - vt)]} [A \cos k_1 \kappa(\mathbf{n} \cdot \mathbf{x} - vt) B \sin k_1 \kappa(\mathbf{n} \cdot \mathbf{x} - vt)], \end{aligned} \quad (5.5)$$

where A, B are constants and κ is defined by

$$\kappa := \sqrt{1 - [(\mathbf{n} \cdot \mathbf{b}) + \frac{\rho v}{2k_1 \nu}]^2}. \quad (5.6)$$

This finite-amplitude wave propagates in the direction of \mathbf{n} with speed v and wave number $k_1 \kappa$, is damped exponentially with respect to time, and is attenuated in the direction of $\mathbf{b} + \sqrt{1 - \kappa^2} \mathbf{n}$. The angle θ (say) between the normal to the planes of constant phase and the normal to the planes of constant amplitude is given by

$$\cot \theta = \mathbf{n} \cdot \mathbf{b} + \sqrt{1 - \kappa^2} = 2(\mathbf{n} \cdot \mathbf{b}) + \frac{\rho v}{k_1 \nu}. \quad (5.7)$$

Note that Boulanger, Hayes, and Rajagopal [15] also investigated the possibility of linearly-polarized, damped, inhomogeneous plane wave solutions to the Navier-Stokes equations. However, they prescribed *a priori* the form of the solution. The result presented above allows for a greater variety of solutions, in the same manner that starting from a finite-amplitude wave in the form (2.10) where f and g are unknown functions, rather than prescribed as exponential and sinusoidal functions, allowed Rodrigues Ferreira and Boulanger [14] to find more solutions than Destrade [16] in the elastic neo-Hookean case.

We also note that the solutions found here for the Navier-Stokes equations are a generalization of the celebrated *Kelvin modes* [17]. These are disturbances in the form of homogeneous plane waves of particular interest in the stability study of basic flows characterized by spatially uniform shearing rates, i.e. self-equilibrated homogeneous motions [18]. This is because Craik and Criminale [19] have shown that the superimposition of a single Kelvin mode to the above mentioned basic flows is an exact solution of the

full Navier-Stokes equations. It would be interesting to investigate if the superimposition of our solutions to homogeneous self-equilibrated motions will give some new exact solutions for the Navier-Stokes equations, but this investigation is outside the framework of the present paper.

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